FOURTH MIDDLE EUROPEAN MATHEMATICAL OLYMPIAD

STREČNO
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Contest Solutions

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Problem I-1. *Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, we have*

$$f(x+y) + f(x)f(y) = f(xy) + (y+1)f(x) + (x+1)f(y).$$

Solution. Setting $y = 0$ yields

$$0 = f(0)(f(x) - x - 2).$$

It is easy to check that $f(x) = x + 2$ *is not* a solution, so $f(0) = 0$. Setting $x = 1$, $y = -1$, we obtain

$$0 = f(-1)(f(1) - 3),$$

so $f(-1) = 0$ or $f(1) = 3$.

Let us assume $f(-1) = 0$. Putting $x = 2$, $y = -1$ in the original functional equation, we get $f(-2) = f(1)$. On the other hand, setting $x = -2$, $y = 1$ gives $f(-2)f(1) = 3f(-2) - f(1)$ which together with $f(-2) = f(1)$ gives $f(1) \in \{0, 2\}$.

So we have $f(1) = a \in \{0, 2, 3\}$. Setting $y = 1$ yields

$$f(x+1) = (3-a)f(x) + a(x+1). \tag{1}$$

for all real x.

Now we set $y = 1 + 1/x$ for arbitrary $x \neq 0$, we obtain

$$f\left(x + \tfrac{1}{x} + 1\right) + f(x)f\left(\tfrac{1}{x} + 1\right) = f(x+1) + \left(\tfrac{1}{x} + 2\right)f(x) + f\left(\tfrac{1}{x} + 1\right)(x+1).$$

Applying (1) yields

$$(3-a)\left(f\left(x + \tfrac{1}{x}\right) + f(x)f\left(\tfrac{1}{x}\right) - (x+1)f\left(\tfrac{1}{x}\right)\right) = f(x)\left(5 - 2a - (a-1)\tfrac{1}{x}\right) + 2a + ax.$$

From this and the original functional equation with $y = 1/x$ we have

$$(3-a)\left(a + f(x)\left(1 + \tfrac{1}{x}\right)\right) = f(x)\left(5 - 2a - (a-1)\cdot\tfrac{1}{x}\right) + 2a + ax,$$

which is equivalent to

$$f(x)\left(-2 + a + \tfrac{2}{x}\right) = a^2 + ax - a.$$

Using $a \in \{0, 2, 3\}$ we get $f(x) = 0$, $f(x) = x^2 + x$, $f(x) = 3x$ for all real x (with some exceptions when $-2 + a + \frac{2}{x} = 0$, but this cases can be handled easily for example by using (1)). We can also easily check that the functions $f(x) = 0$, $f(x) = x^2 + x$, $f(x) = 3x$ are solutions of the original functional equation.

Solution 2. As in the first solution we obtain $f(0) = 0$, $f(1) = a \in \{0, 2, 3\}$ and

$$f(x+1) = (3-a)f(x) + a(x+1) \quad \text{for } x \in \mathbb{R}. \tag{2}$$

If $a = 3$, then (2) implies that $f(x+1) = 3(x+1)$ for all $x \in \mathbb{R}$ (and therefore $f(x) = 3x$ for all x). It is easy to verify that this is a solution of the functional equation.

In the sequel we will assume that $a \in \{0, 2\}$ and therefore $f(-1) = 0$. We will compute $f(xyz)$ in two different ways. Setting yz for y into the original functional equation we obtain

$$\begin{aligned} f(xyz) &= f(x+yz) + f(x)f(yz) - (yz+1)f(x) - (x+1)f(yz) = \\ &= f(x+yz) + f(x)f(y+z) + f(x)f(y)f(z) - (z+1)f(x)f(y) - \\ &\quad - (y+1)f(x)f(z) - (yz+1)f(x) - (x+1)f(y+z) - (x+1)f(y)f(z) + \\ &\quad + (x+1)(z+1)f(y) + (x+1)(y+1)f(z). \end{aligned}$$

On the other hand, setting xy for x and z for y into the original functional equation we obtain

$$\begin{aligned} f(xyz) &= f(xy+z) + f(xy)f(z) - (z+1)f(xy) - (xy+1)f(z) = \\ &= f(xy+z) + f(x+y)f(z) + f(x)f(y)f(z) - (y+1)f(x)f(z) - \\ &\quad - (x+1)f(y)f(z) - (z+1)f(x+y) - (z+1)f(x)f(y) + (y+1)(z+1)f(x) + \\ &\quad + (x+1)(z+1)f(y) - (xy+1)f(z). \end{aligned}$$

Therefore

$$\begin{aligned} &f(x+yz) + f(x)f(y+z) - (yz+1)f(x) - (x+1)f(y+z) + (x+1)(y+1)f(z) = \\ &= f(xy+z) + f(x+y)f(z) - (z+1)f(x+y) + (y+1)(z+1)f(x) - (xy+1)f(z). \end{aligned}$$

In particular, for $x = -1$ and $y = z$ we obtain

$$f(z^2-1) = f(z-1)f(z) - (z+1)f(z-1) + (z-1)f(z).$$

On the other hand, setting $x = z+1$ and $y = z-1$ into the original equation we obtain

$$f(2z) + f(z+1)f(z-1) = f(z^2-1) + (z+2)f(z-1) + zf(z+1).$$

Therefore

$$f(2z) + f(z+1)f(z-1) = f(z-1)f(z) + f(z-1) + (z-1)f(z) + zf(z+1).$$

Since (2) implies that $f(2) = 5a - a^2$ and since for $x = 2$ and $y = z$ the original functional equation implies

$$f(z+2) + f(2)f(z) = f(2z) + (z+1)f(2) + 3f(z),$$

it follows that

$$\begin{aligned} &f(z+2) + (5a-a^2)f(z) + f(z+1)f(z-1) = \\ &= f(z-1)f(z) + f(z-1) + (z+2)f(z) + zf(z+1) + (5a-a^2)(z+1). \end{aligned}$$

The equation (2) implies that

$$\begin{aligned} f(z) &= (3-a)f(z-1) + az, \\ f(z+1) &= (3-a)^2 f(z-1) + (4a-a^2)z + a \end{aligned}$$

and

$$f(z+2) = (3-a)^3 f(z-1) + (a^3 - 7a^2 + 13a)z + 5a - a^2.$$

From the last four equations we obtain that

$$\begin{aligned}(3-a)(2-a)f(z-1)^2 - 2(a-3)(a-2)zf(z-1) + (a^2-9a+20)f(z-1) = \\ = (5a-a^2)z^2 + (a^2-5a)z.\end{aligned}$$

For $a = 2$ we get $6f(z-1) = 6z^2 - 6z$, therefore $f(z) = z^2 + z$ and it is easy to verify that this is a solution of the functional equation.

For $a = 0$ we get $6f(z-1)^2 - 12zf(z-1) + 20f(z-1) = 0$, which implies that for each $z \in \mathbb{R}$ one of the equalities $f(z) = 0$ and $f(z) = 2z - \frac{4}{3}$ is satisfied. Assume that $f(z) = 2z - \frac{4}{3}$ for some $z \in \mathbb{R}$. Then the original functional equation for $x = 1$ and $y = z$ implies that $f(z+1) = 3f(z) = 6z - 4$. Therefore either $6z - 4 = 0$ or $6z - 4 = 2(z+1) - \frac{4}{3}$. The first equation implies that $z = \frac{2}{3}$ and $f(z) = 2z - \frac{4}{3} = 0$, and the second equation implies that $z = \frac{7}{6}$ and therefore $f(z) = 2z - \frac{4}{3} = 1$, $f(z+1) = 3z = 3 = 2(z+1) - \frac{4}{3}$ and $f(z+2) = 3f(z+1) = 9 \neq 5 = 2(z+2) - \frac{4}{3}$. The contradiction shows that $f(z) = 0$ for each $z \in \mathbb{R}$.

Problem I-2. *All positive divisors of a positive integer N are written on a blackboard. Two players A and B play the following game taking alternate moves. In the first move, the player A erases N. If the last erased number is d, then the next player erases either a divisor of d or a multiple of d. The player who cannot make a move loses. Determine all numbers N for which A can win independently of the moves of B.*

Solution. Let $N = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ be the prime factorization of N. In an arbitrary move the players writes down a divisor of N, which we can represent as a sequence $(b_1, b_2, \dots, b_k)$, where $b_i \le a_i$ (such a sequence represents the number $p_1^{b_1} p_2^{b_2} \dots p_k^{b_k}$). The rules of the game say that the sequence $(b_1, b_2, \dots, b_k)$ can be followed by a sequence $(c_1, c_2, \dots, c_k)$ with either $c_i \le b_i$ for each i, or $a_i \ge c_i \ge b_i$ for each i (obviously, if such a sequence is not on the sheet).

If one of the numbers a_i is odd, then the player B posses the winning strategy. Indeed, let for simplicity a_1 be odd. Then the response for the move $(b_1, b_2, \dots, b_k)$ should be

$$(a_1 - b_1, b_2, \dots, b_k).$$

One can easily check that this is a winning strategy for B: All the legal sequences split up into pairs and when A writes down one sequence from a pair, player B responds with the second one from the same pair ($a_1 - b_1 \ne b_1$ because of a_1 is odd).

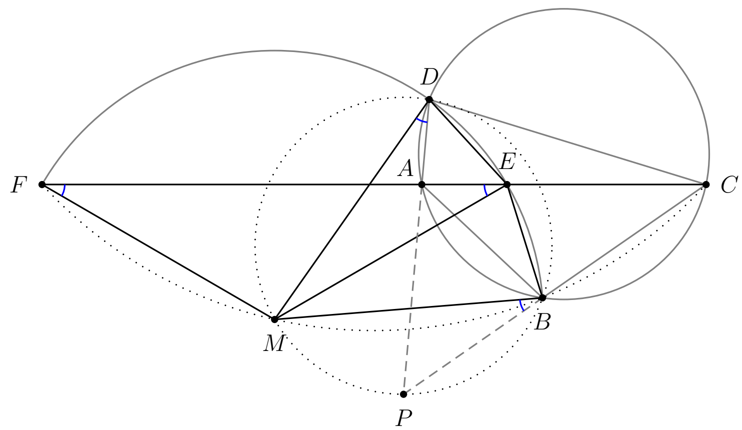
If all a_i are even, then the player A has a winning strategy. Let the move of player B be $(b_1, b_2, \dots, b_k)$, where one of b_i is strictly less than a_i ($(b_1, b_2, \dots, b_k) \ne (a_1, a_2, \dots, a_k)$, as it was the first move of A). Let j be the smallest index such that $b_j < a_j$. Then the response of A can be

$$(b_1, b_2, \dots, b_{j-1}, a_j - b_j - 1, b_{j+1}, \dots, b_k) \qquad \text{(symmetric reflection of } b_j\text{)}.$$

Again, all legal sequences (except for $(a_1, a_2, \dots, a_k)$) split up into pairs and when B writes down one sequence from a pair, player A can respond with the second one ($a_j - b_j - 1 \ne b_j$ because of a_j is even).

Obviously the condition "all a_i are even" means that "N is a square". For $N \in [2000, 2100]$ it is possible only for $N = 2025$. The answer for the alternative question is 2010^{1005}.

Problem I-3. *We are given a cyclic quadrilateral $ABCD$ with a point E on the diagonal AC such that $AD = AE$ and $CB = CE$. Let M be the center of the circumcircle k of the triangle BDE. The circle k intersects the line AC in the points E and F. Prove that the lines FM, AD, and BC meet at one point.*



Solution. Assume that A lies between C and F (the case when C lies between A and F can be handled in the same way). Let the lines BC and AD meet in P. Using $MB = ME$, $BC = CE$ and $ME = MF$ we get $\triangle MBC \cong \triangle MEC$ and therefore

$$\angle MBC = \angle MEC = 180^\circ - \angle MEF = 180^\circ - \angle MFC,$$

which implies that the points M, B, C, and F are concyclic. Using $ME = MD$ and $AE = AD$ (which means $\triangle MEA \cong \triangle MDA$) we get $\angle AEM = \angle ADM$, hence $\angle MDP = \angle MBP$ and the quadrilateral $MPBD$ is cyclic. This gives (using that quadrilaterals $ABCD$ and $FMBC$ are cyclic as well) that

$$\angle PMB = \angle PDB = \angle ADB = \angle ACB = \angle FCB = 180^\circ - \angle FMB,$$

which means that the points F, M and P are collinear, therefore the lines AD, BC and FM meet in a point.

Solution 2. Like in the first solution, $FMBC$ is cyclic. Since the points M and A lie on the perpendicular bisector of the segment DE, we have $\angle MDA = \angle MEA$. Moreover, $ME = MF$, hence $\angle MFA = \angle MFE = \angle MEF = \angle MEA$. Therefore, the quadrilateral $MADF$ is cyclic. Then the radical axes of circumcircles of cyclic quadrilaterals $FMBC$, $BCDA$ and $ADFM$ are the lines FM, AD and BC, which meet in the radical center of the three circles. Thus, the lines FM, AD and BC meet in a point.

Problem I-4. *Find all positive integers n which satisfy the following two conditions:*

(i) n has at least four different positive divisors;

(ii) for any divisors a and b of n satisfying $1 < a < b < n$, the number $b - a$ divides n.

Solution. Clearly primes, squares of primes and the number 1 have the given property. We will exclude these numbers from further considerations.

First assume that n is even; thus $n = 2x$ for some integer x. Then $x - 2$ divides n. Any divisor of n smaller then $x = n/2$ is at most $n/3$. Therefore, $x - 2 \leq n/3$, yielding $x \leq 6$. Checking all possibilities for x, we arrive to three new satisfactory values of n, namely, 6, 8, and 12.

Next assume that n is odd. Let $n = px$, where p is the smallest divisor of n greater than one. Obviously p is an odd prime. Note that $p + 1 \nmid n$ because $p + 1$ is even; thus $x \neq p + 1$. Since $1 < p < x < n$, we have

$$x - p \mid px.$$

If $p \nmid x$ then $x - p$ and x are coprime; hence $x - p \mid p$. Then $x - p \leq p$. On the other hand, $x \neq p + 1$; therefore, $x - p \geq p$ since p is the smallest nontrivial divisor. Hence $x = 2p$, contradicting the assumption that n is odd.

If $p \mid x$, then $x = py$ for some integer y greater than one. The choice of p implies that $y \geq p$. Moreover, $py - p \mid p^2 y$; thus $y - 1 \mid py$. Since $y - 1$ and y are coprime, $y \leq p + 1$. If $y = p + 1$, then y is an even divisor of n; this contradicts the assumption that n is odd. Otherwise $y = p$ since $y \geq p$. This contradicts the condition $y - 1 \mid py$.

All the solutions are primes, squares of primes and the numbers 1, 6, 8 and 12.

Solution 2. If $n = 1$, n is prime or n is a square of a prime, there are no divisors a and b satisfying $n > a > b > 1$, thus the condition is trivially satisfied. Otherwise, for some integers k and ℓ, $n = \ell \cdot k$ with $\ell > k > 1$. Then $\ell - k$ also divides $n = \ell \cdot k$.

If ℓ and k are coprime, $\ell - k$ is coprime to ℓ and k, thus $\ell - k = 1$. Hence $n = k(k+1)$. Let p be a prime divisor of k. Since $k + 1 - p$ is coprime to $p(k+1)$, the condition implies that $k + 1 - p$ divides k. But

$$k + 1 - p = (p - 1)\left(\frac{k}{p} - 1\right) + \frac{k}{p}$$

divides k if and only if $(p-1)(k/p - 1) = 0$; thus $k = p$ and $n = p(p+1)$. Clearly $p = 2$ gives a solution $n = 6$. Otherwise $p + 1 = q \cdot r$ for some prime q and integer r greater than 1. Since

$$p - q = qr - 1 - q = (q-1)(r-1) - 2 + r$$

is a divisor of r, we have $(q-1)(r-1) \leq 2$. This gives only three possibilities: $q = r = 2$ or $q = 2$, $r = 3$ or $q = 3$, $r = 2$. The first one yields a solution $n = 12$, while the other two give $n = 30$, which fails to satisfy the conditions: $6 - 2 \nmid 30$.

It remains to consider the case when n cannot be written as a product of two coprime numbers greater than 1. Then $n = p^a$, where $a \geq 3$ (for $a \leq 2$, we obtain the solutions we have already described). This implies that p and p^2 are proper divisors of n, hence $p^2-p=p(p-1)$ divides $n=p^a$. Since p and $p-1$ are coprime, this is only possible when $p-1=1$; thus $p=2$. However, $2^3-2=6$ is not a divisor of 2^a; hence there are no solutions for $a \geq 4$. Only the number 8 satisfies the condition in this case.

Solution 3. Clearly 1, p, p^2 are solutions. For the other prime powers, $n=p^k$ is possible only for $p=2$ and $k<4$ (p^2-p is even for odd p, $8-2=6$ does not divide 16).

Now, n is not a prime power, then it has two (or more) prime factors p, q. Then $q-p \mid n$. If p, q are both odd, then $2 \mid n$. (Otherwise also $2 \mid n$.) Therefore, if n is not a prime power, one of its factors is 2.

Let p be the smallest divisor of n larger than 2; $1<2<p<n$. From $p-2 \mid n$ follows $p=3$, so $6 \mid n$. Clearly, $n=6$ is a solution.

Let $n=6a$, $a>1$. Then $1<3<3a<6a=n$, therefore

$$(3a-3)=3(a-1) \mid n=6a,$$
$$a-1 \mid 2a.$$

Since $\gcd(a, a-1)=1$, we have $a-1 \mid 2$, hence $a-1=1$ or $a-1=2$, which yields $n=12$ or $n=18$.

It is easy to check that $n=12$ is a solution, and $n=18$ is not (e. g., $7=9-2$ is not a divisor of 18).

Problem T-1.

Three strictly increasing sequences

$$a_1, a_2, a_3, \dots, \qquad b_1, b_2, b_3, \dots, \qquad c_1, c_2, c_3, \dots$$

of positive integers are given. Every positive integer belongs to exactly one of the three sequences. For every positive integer n, the following conditions hold:

(i) $c_{a_n} = b_n + 1$;

(ii) $a_{n+1} > b_n$;

(iii) the number $c_{n+1}c_n - (n+1)c_{n+1} - nc_n$ *is even.*

Find a_{2010}, b_{2010}, *and* c_{2010}.

Solution. Since $\{c_n\}$ is a strictly increasing sequence of positive integers, it is clear that $c_n \geq n$, $n \in \mathbb{N}$. Hence, $c_{a_n} \geq a_n$, $n \in \mathbb{N}$. However, the given sequences do not contain equal terms, so $c_{a_n} > a_n$ and $b_n = c_{a_n} - 1 > a_n$, $n \in \mathbb{N}$. Similarly, from (ii) and (iii), $a_{n+1} > b_n + 1 = c_{a_n}$, $n \in \mathbb{N}$. It is also easy to see that $b_n < c_{a_n} < b_{n+1}$. Let us for any $n \in \mathbb{N}$ count the number of terms in all three sequences that are less or equal to c_{a_n}. There are n such terms in the first sequence (that is, $a_1, a_2, \dots, a_n$), n such terms in the second sequence $(b_1, b_2, \dots, b_n)$ and a_n such terms in the third sequence $(c_1, c_2, \dots, c_{a_n})$. It is $2n + a_n$ terms in total. By (i) any positive integer less or equal c_{a_n} must appear among these terms exactly once; thus, the total number of these terms equals

$$2n + a_n = c_{a_n}. \tag{1}$$

Now we take a_n instead of n in (iv):

$$c_{a_n+1}c_{a_n} - (a_n+1)c_{a_n+1} - a_n c_{a_n} = c_{a_n+1}(a_n + 2n) - (a_n+1)c_{a_n+1} - a_n(a_n+2n) =$$
$$= c_{a_n+1}(2n-1) - a_n^2 - 2a_n n \equiv c_{a_n+1} - a_n - 2n \equiv c_{a_n+1} - c_{a_n} \equiv 0 \pmod 2.$$

This means that $c_{a_n+1} \neq c_{a_n} + 1$ and the number $c_{a_n} + 1$ has to belong to either of the first two sequences. The inequalities $a_n < b_n < c_{a_n} < a_{n+1} < b_{n+1}$ imply that $c_{a_n} + 1 = a_{n+1}$, $n \in \mathbb{N}$, and, by (1),

$$a_{n+1} = a_n + 2n + 1, \quad n \in \mathbb{N}. \tag{2}$$

Next we prove that $a_1 = 1$. Indeed, number 1 has to belong to one of the given sequences, and if $a_1 > 1$ then $c_1 = 1$ or $b_1 = 1$. The latter case is impossible because $b_1 > a_1$. Then we must have $c_1 = 1$, and either $c_2 = 2$, or $a_1 = 2$ and $c_{a_1} = c_2 = a_1 + 2 = 4$. In both cases we obtain a contradiction by setting $n = 1$ in (iv). This proves that $a_1 = 1$, and, together with (2), defines a unique sequence $\{a_n\}$:

$$a_n = a_{n-1} + (2n-1) = a_{n-2} + (2n-3) + (2n-1) = \dots = a_1 + 3 + 5 + \dots + (2n-1) = n^2, \quad n \in \mathbb{N}.$$

Hence,

$$a_{2010} = 2010^2,$$
$$b_{2010} = c_{a_{2010}} - 1 = a_{2010} + 2 \cdot 2010 - 1 = 2011^2 - 2,$$
$$c_{1936} = c_{44^2} = c_{a_{44}} = a_{44} + 2 \cdot 44 = 44^2 + 88 = 2024,$$
$$a_{45} = 45^2 = 2025,$$

and all the integers between a_{45} and $b_{45} = c_{a_{45}} - 1 = a_{45} + 2 \cdot 45 - 1 = a_{45} + 89$ belong to the sequence $\{c_n\}$. Hence, these integers have the form

$$c_{1936+k} = a_{45} + k, \quad k = 1, 2, ..., 88,$$

and $c_{2010} = c_{1936+74} = a_{45} + 74 = 2099$.

Answer. $a_{2010} = 2010^2$, $b_{2010} = 2011^2 - 2$, $c_{2010} = 2099$.

Solution 2. Denote by $(*)$ the trivial fact $a_n < c_{a_n}$ derived at the beginning of the first solution. One can easily fill the sequences inductively. In fact, like in the first solution, we have $a_1 = 1$. Now we will find the place for number 2. If $a_2 = 2$, then by (iii) $2 = a_2 > b_1$, which is impossible. If $c_1 = 2$, then by (ii) we have $2 = c_1 = c_{a_1} = b_1 + 1$, hence $b_1 = 1$ which is also impossible. So the only way is to put $b_2 = 2$. Then by (ii) $c_1 = c_{a_1} = b_1 + 1 = 3$.

n	1	2	3	4	5	...
a_n	1					
b_n	2					
c_n	3					

Now, because of (iv), we have $c_2 \neq 4$. Also, $b_2 \neq 4$, because otherwise by $(*)$ and (ii) $a_2 < c_{a_2} = b_2 + 1 = 5$ and there is no number left for a_2. So we have $a_2 = 4$. Then by (iii) $a_3 \neq 5$. Also, $b_2 \neq 5$, because otherwise by (ii) $c_4 = c_{a_2} = b_2 + 1 = 6$ and there are no numbers left for c_2, c_3. So we have $c_2 = 5$. Using the same arguments we derive $a_3 \neq 6$ and $b_2 \neq 6$, hence, $c_3 = 6$. Now, $a_3 \neq 7$ (by (iii)). Also, $c_4 \neq 7$, because otherwise by (ii) $7 = c_4 = c_{a_2} = b_2 + 1$, and this leads to $b_2 = 6$, which is not true. Hence, $b_2 = 7$. Then $c_4 = c_{a_2} = b_2 + 1 = 8$.

n	1	2	3	4	5	...
a_n	1	4				
b_n	2	7				
c_n	3	5	6	8		

Now, we can repeat the arguments from the last paragraph: Because of (iv) we have $c_5 \neq 9$. By $(*)$ and (ii) we have $b_3 \neq 9$ (otherwise $a_3 < c_{a_3} = b_3 + 1 = 10$ and there is no number left for a_3). So we have $a_3 = 9$. By (iii), $a_4 \neq 10$. By (ii), $c_9 = c_{a_3} = b_3 + 1$, therefore $b_3 \neq 10$ (otherwise there are no numbers left for $c_5, \ldots, c_8$). So we have $c_5 = 10$. Similarly

$$\begin{aligned} a_4 \neq 11,\ b_3 \neq 11 \quad &\Longrightarrow \quad c_6 = 11, \\ a_4 \neq 12,\ b_3 \neq 12 \quad &\Longrightarrow \quad c_7 = 12, \\ a_4 \neq 13,\ b_3 \neq 13 \quad &\Longrightarrow \quad c_8 = 13. \end{aligned}$$

Finally, $a_4 \neq 14$ (by (iii)), $c_9 \neq 14$ (otherwise by (ii) $14 = c_9 = c_{a_3} = b_3 + 1$, and this leads to $b_3 = 13$, which is not true). Hence, $b_3 = 14$ and $c_9 = c_{a_3} = b_3 + 1 = 15$.

n	1	2	3	4	5	6	7	8	9	...
a_n	1	4	9							
b_n	2	7	14							
c_n	3	5	6	8	10	11	12	13	15	

We formulate the claim which can be easily proved by induction. (We will skip the formal proof. However, it is just an obvious generalization of the last two paragraphs.) For $k \in \mathbb{N}$ and $i = 1, 2, \ldots, 2k-2$, we have

$$\begin{aligned}
a_k &= k^2, \\
b_k &= k^2 + 2k - 1, \\
c_{(k-1)^2+i} &= k^2 + i, \\
c_{k^2} &= k^2 + 2k.
\end{aligned}$$

The rest is straightforward:

$$a_{2010} = 2010^2, \quad b_{2010} = 2010^2 + 2 \cdot 2010 - 1, \quad c_{2010} = c_{44^2+74} = 45^2 + 74 = 2099.$$

Problem T-2. *For each integer $n \geq 2$, determine the largest real constant C_n such that for all positive real numbers $a_1, \ldots, a_n$, we have*

$$\frac{a_1^2 + \cdots + a_n^2}{n} \geq \left(\frac{a_1 + \cdots + a_n}{n}\right)^2 + C_n \cdot (a_1 - a_n)^2.$$

Solution. Define $x_{ij} = a_i - a_j$ for $1 \leq i < j \leq n$. After multiplication with n^2, the difference of the squares of quadratic and arithmetic mean equals

$$n^2(\mathrm{QM}^2 - \mathrm{AM}^2) = n(a_1^2 + \cdots + a_n^2) - (a_1 + \cdots + a_n)^2 = (n-1)\sum_{i=1}^{n} a_i^2 - \sum_{i<j} 2a_i a_j =$$

$$= \sum_{i<j} x_{ij}^2 = x_{1n}^2 + \sum_{i=2}^{n-1}(x_{1i}^2 + x_{in}^2) + \sum_{1<i<j<n} x_{ij}^2.$$

Here the last sum is clearly non-negative. By the AM-QM inequality the sum in the middle is at least

$$\sum_{i=2}^{n-1}(x_{1i}^2 + x_{in}^2) \geq \frac{1}{2}\sum_{i=2}^{n-1}(x_{1i} + x_{in})^2 = \frac{n-2}{2} \cdot x_{1n}^2.$$

Hence we finally get

$$n^2(\mathrm{QM}^2 - \mathrm{AM}^2) \geq x_{1n}^2 + \frac{n-2}{2} \cdot x_{1n}^2 = \frac{n}{2} \cdot (a_1 - a_n)^2$$

with equality if and only if $a_2 = \cdots = a_{n-1} = \frac{1}{2}(a_1 + a_n)$. The largest such constant is therefore

$$C = \frac{1}{2n}.$$

Problem T-3. *In each vertex of a regular n-gon there is a fortress. At the same moment each fortress shoots at one of the two nearest fortresses and hits it. The result of the shooting is the set of the hit fortresses; we do not distinguish whether a fortress was hit once or twice. Let $P(n)$ be the number of possible results of the shooting. Prove that for every positive integer $k \geq 3$, $P(k)$ and $P(k+1)$ are relatively prime.*

Solution. Let us denote each hit fortress by a black dot and each undamaged one with a white dot. Then $P(n)$ is the number of colourings of n dots distributed on the circle with black and white colours in such a way, that no two white dots have exactly one dot in between them. The proof of this bijectivity is straightforward: If there are two white dots with exactly one dot in between, then obviously the fortress in between can not shoot, which is not permitted. On the other hand, if there are no such two white dots, then each fortress can shoot at least one black dot and to ensure that every black dot will be hit, we can force the one in the clockwise direction to shoot at it.

If n is odd, then $P(n)$ is equal to the number $K(n)$ of colourings of n dots on a circle with black and white colours in such a way, that no two neighbouring dots have white colour (we define the neighbouring dots to be the dots which have exactly one other dot in between them). For n even, with the same definition of neighbours, the circle splits into two circles with $n/2$ dots, and we have $P(n) = K(n/2)^2$.

For $K(n)$ it is easy to derive a recurrence formula $K(n) = K(n-1) + K(n-2)$. In fact, the number of legal colourings with n-th dot being black is equal to the number of legal colourings of $n-1$ dots (just put the black dot in between the first dot and the $(n-1)$-th dot) plus the number of colourings of $n-1$ dots with no two neighbouring white dots except for the first and $(n-1)$-th (we can put the black dot in between two white dots to obtain legal colouring). The latter case gives the same number as the number of legal colouring with $n-2$ dots having the first dot white (just span two white dots into one white). On the other hand, the number of legal colourings with n-th dot being white is equal to the number of colourings of $n-1$ dots with no two neighbouring white dots and with the first and $(n-1)$-th dot black (we can put the white dot only in between to black dots), which is equal to the number of legal colouring with $n-2$ dots having the first dot black (again, span two black dots into one black). Together, we have

$$K(n) = K(n-1) + K_w(n-2) + K_b(n-2) = K(n-1) + K(n-2),$$

where K_w and K_b stands for the number of legal colourings with first dot white and black respectively.

Moreover we can directly count $K(2) = 3$, $K(3) = 4$, $K(4) = 7$, which suggests

$$K(2) = F(4) - F(0), \qquad K(3) = F(5) - F(1), \qquad K(4) = F(6) - F(2)$$

and we can easily prove by the induction $K(n) = F(n+2) - F(n-2)$, where $F(k)$ stands for the k-th term of the Fibonacci sequence ($F(0) = 0$, $F(1) = F(2) = 1, \dots$). Further $(K(2), K(3)) = 1$, and for $n \geq 3$ we have

$$(K(n), K(n-1)) = (K(n) - K(n-1), K(n-1)) = (K(n-2), K(n-1)) = \cdots = 1.$$

Similarly we show that for each even $n = 2a$ the number $P(n) = K(a)^2$ is relatively prime both to $P(n+1) = K(2a+1)$ and $P(n-1) = K(2a-1)$:

$$\begin{aligned}(K(a), K(2a+1)) &= (K(a), F(2)K(2a) + F(1)K(2a-1)) = \\ &= (K(a), F(3)K(2a-1) + F(2)K(2a-2)) = \dots \\ &\dots = (K(a), F(a+1)K(a+1) + F(a)K(a)) = (K(a), F(a+1)) = \\ &= (F(a+2) - F(a-2), F(a+1)) = \\ &= (F(a+2) - F(a+1) - F(a-2), F(a+1)) = \\ &= (F(a) - F(a-2), F(a+1)) = (F(a-1), F(a+1)) = \\ &= (F(a-1), F(a)) = 1\end{aligned}$$

$$\begin{aligned}(K(a), K(2a-1)) &= (K(a), F(2)K(2a-2) + F(1)K(2a-3)) = \\ &= (K(a), F(3)K(2a-3) + F(2)K(2a-4)) = \dots \\ &\dots = (K(a), F(a)K(a) + F(a-1)K(a-1)) = (K(a), F(a-1)) = \\ &= (F(a+2) - F(a-2), F(a-1)) = (F(a+2) - F(a), F(a-1)) = \\ &= (F(a+2) - F(a+1), F(a-1)) = (F(a), F(a-1)) = 1,\end{aligned}$$

which finishes the proof.

Problem T-4. *Let n be a positive integer. A square $ABCD$ is partitioned into n^2 unit squares. Each of them is divided into two triangles by the diagonal parallel to BD. Some of the vertices of the unit squares are colored red in such a way that each of these $2n^2$ triangles contains at least one red vertex. Find the least number of red vertices.*

Solution. The least number of red vertices is

$$\left\lfloor \frac{(n+1)^2}{3} \right\rfloor .$$

First, we define a colouring and count the number of red vertices. In what follows it will be shown that the number of red vertices, obtained by this colouring, is indeed minimal.

It is convenient to replace the square by rhombus $ABCD$ as in this case the isosceles rightangled triangles are equilateral. Cover the rhombus by regular unit hexagons in such a way, that A lies in the vertex of a hexagon (figure 1). Colour the center of each hexagon red. Clearly each equilateral unit triangle belongs to one of the hexagons and thus contains a red point. Hence this colouring satisfies the required conditions.

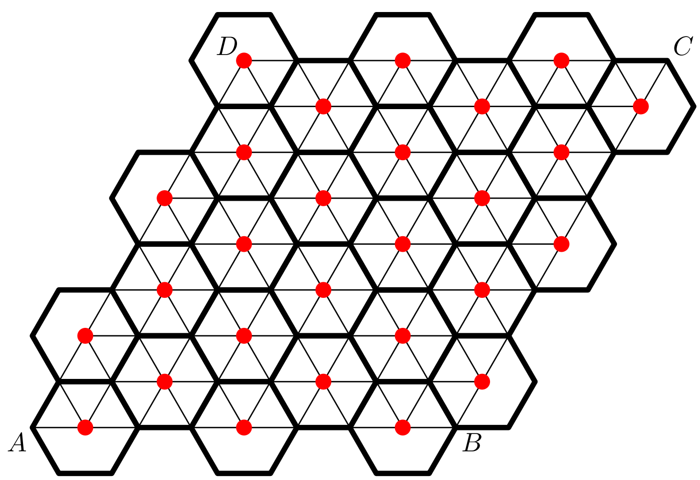


Figure 1: Covering with regular hexagons.

Denote by a_n the number of vertices which are coloured red. Let $A_1, A_2, \dots, A_{n-1}$ be the points on AB such that $AA_1 = A_1A_2 = \dots = A_{n-2}A_{n-1} = A_{n-1}B = 1$. Similarly define points $B_1, \dots, B_{n-1}$ on BC, points $C_1, \dots, C_{n-1}$ on CD, and $D_1, \dots, D_{n-1}$ on DA.

Each of the n points on the line A_1B_{n-1} is red (figure 1 and 2). The parallel lines A_2B_{n-2} and A_3B_{n-3} have no red points while line A_4B_{n-4} contains 3 points less than A_1B_{n-1}, that is $n-3$. All of them are red. Similarly, red points lie on lines A_7B_{n-7}, $A_{10}B_{n-10}$ and so on. The number of red points each time decreases by 3. On the other side of the diagonal AC, we have $n-1$ red points on the line C_2D_{n-2}, $n-4$ points on C_5D_{n-5} and so on. Hence the number of red points equals

$$a_n = \big(n + (n-3) + (n-6) + \dots\big) + \big((n-1) + (n-4) + (n-7) + \dots\big).$$

When n is of the form $n = 3k+1$, we have $a_n = \frac{1}{3}n(n+2)$, for $n = 3k+2$ we get $a_n = \frac{1}{3}(n+1)^2$ and $n = 3k+3$ implies $a_n = \frac{1}{3}n(n+2)$. In general, $a_n = \lfloor \frac{1}{3}(n+1)^2 \rfloor$.

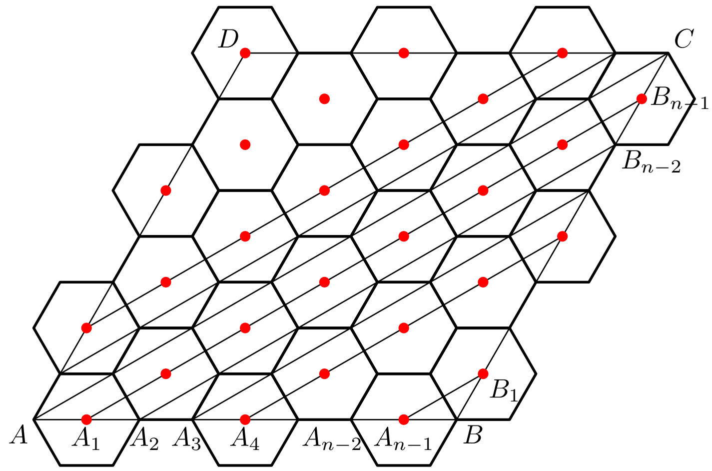


Figure 2: The value of a_n.

Let b_n denote the least number of red points needed. Clearly, $b_1 = 1$. Consider a triangle with side length 2 (the first picture on figure 3). The four unit triangles do not have a common vertex, hence at least two vertices must be coloured. Each of the small marked disjoint triangles in the second and third picture (figure 3) must contain at least one red vertex and the bigger marked triangle at least two. This implies that $b_2 \geq 2 + 1 = 3$ and $b_3 \geq 1 + 1 + 1 + 2 = 5$.

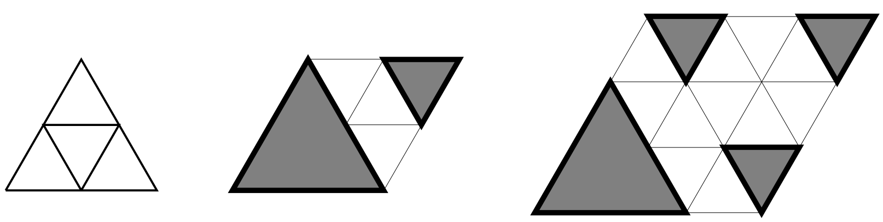


Figure 3: The minimum number of red vertices for $n = 2$ and $n = 3$.

It has thus been shown that for $n = 1$, 2, and 3, $b_n \geq a_n$, which means that $b_n = a_n$. The rest will follow by induction. The next step will show that if $b_{n-3} = a_{n-3}$, then $b_n = a_n$.

Let $n = 3k + 2$. As demonstrated in figure 4, at least $b_{n-3} + (2k + 1)b_2$ red vertices are needed. That is,

$$b_n \geq b_{n-3} + (2k+1) \cdot 3 = \left\lfloor \frac{(n-2)^2}{3} \right\rfloor + 2n - 1 = \left\lfloor \frac{(n+1)^2}{3} \right\rfloor = a_n,$$

hence $b_n = a_n$.

If $n = 3k + 3$, we can estimate that

$$b_n \geq b_{n-3} + 2kb_2 + 2 + 1 + 1 + 1 = \left\lfloor \frac{(n-2)^2}{3} \right\rfloor + 2(n-3) + 5 = \left\lfloor \frac{(n+1)^2}{3} \right\rfloor = a_n.$$

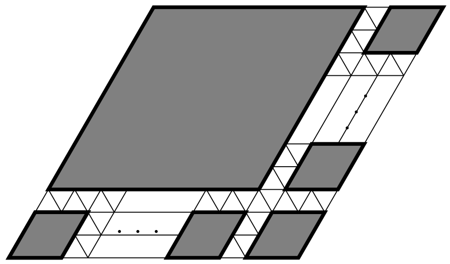


Figure 4: Case $n = 3k + 2$.

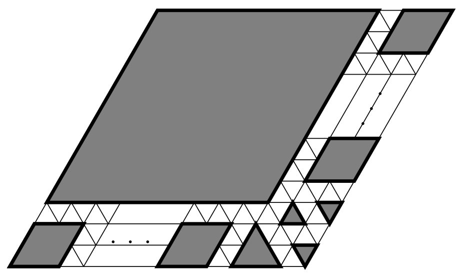


Figure 5: Case $n = 3k + 3$.

Finally, taking $n = 3k + 1$, the last picture demonstrates that

$$b_n \geq b_{n-3} + (2(k-1)+1)b_2 + 1 + 1 + 1 + 1 = \left\lfloor \frac{(n-2)^2}{3} \right\rfloor + 2n - 1 = \left\lfloor \frac{(n+1)^2}{3} \right\rfloor = a_n.$$

In all the cases it follows that $b_n = a_n$.

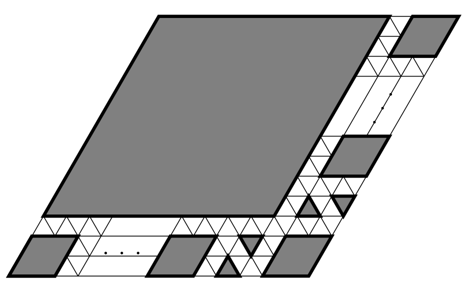
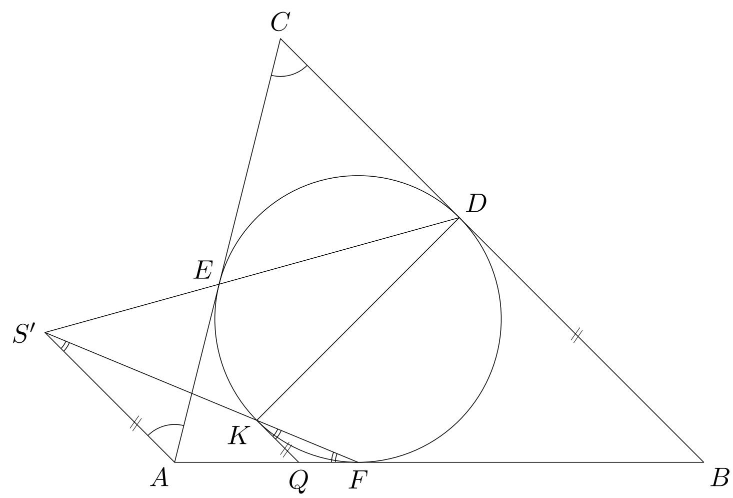


Figure 6: Case $n = 3k + 1$.

Problem T-5. *The incircle of the triangle ABC touches the sides BC, CA, and AB in the points D, E, and F, respectively. Let K be the point symmetric to D with respect to the incenter. The lines DE and FK intersect at S. Prove that AS is parallel to BC.*

Solution. Let S' be the intersection point of the line FK and the line parallel to BC passing through A. We need to show that S', D and E are collinear. Let the tangent line to the given circle at the point K intersect AB at Q (it is parallel to BC). Then $\angle AS'F = \angle QKF = \angle QFK$ and from that follows that $AS' = AF = AE$. But $DC = EC$ and $BC \parallel AS'$. Thus $\angle CDE = \angle CED = \angle AES' = \angle AS'E = \angle AES$ and S', D and E are collinear.



Solution 2. Let $\alpha = \angle BAC$, $\beta = \angle ABC$ and let I be the center by the incircle. Then $\angle IDF = \angle IFD = \beta/2 = \angle AFS$ since $\angle KFD = \angle AFI$. Because $\angle FDS = \angle FIE/2 = 90^\circ - \alpha/2$ ($AFIE$ is cyclic) and $\angle AIF = 90^\circ - \alpha/2$ we have $\triangle AFI \sim \triangle SFD$. The ratio of similitude gives $AF : SF = IF : DF$ and using $\angle AFS = \angle IFD$ yields to $\triangle AFS \sim \triangle IFD$ which means that $AF = AS = AE$. Finally $\triangle ASE \sim \triangle CDE$ gives $\angle SAE = 180^\circ - \alpha - \beta$ and consequently $\angle BAS + \angle ABC = 180^\circ$.

Solution 3. Let $\alpha = \angle BAC$, then $\angle FIE = 180^\circ - \alpha$, $\angle FDE = 90^\circ - \alpha/2$. Because KD is a diameter of the incircle we have $\angle DSF = \alpha/2$ and $\angle KED = 90^\circ$. If $\gamma = \angle BCA$ then $\angle KDE = \gamma/2$. We want to prove that $AE : EC = SE : ED$ because then $\angle CDE = \angle CED = \angle AES = \angle ASE$ which implies $AS \parallel CD$. Calculation of these length in terms of the angles of the triangle ABC and its inradius gives

$$AE = r\cot\frac{\alpha}{2}$$
$$EC = r\cot\frac{\gamma}{2}$$
$$SE = EK\cot\frac{\alpha}{2} = ED\tan\frac{\gamma}{2}\cdot\cot\frac{\alpha}{2}$$

and we are done.

Solution 4. We will use complex coordinates. Let $I = (0)$, and let the incircle be a unit circle. Let $D = (-i)$, $E = (e)$, $F = (f)$. Then $K = (i)$. The tangents in E, F are ($\bar{w}$ is the complex conjugate of w)

$$t_e\colon \qquad z + e^2\bar{z} = 2e,$$
$$t_f\colon \qquad z + f^2\bar{z} = 2f$$

and the coordinates of A (intersection of t_e, t_f)

$$a = \frac{2ef}{e+f}, \qquad \bar{a} = \frac{2}{e+f}.$$

The lines

$$DE\colon \qquad z - ie\bar{z} = e - i,$$
$$FK\colon \qquad z + if\bar{z} = f + i$$

intersect in $S = (z)$,

$$\bar{z} = \frac{(-i)(f-e+2i)}{e+f}, \qquad z = \frac{i(e-f-2ief)}{e+f}.$$

To prove $AS \parallel BC$, we calculate the slope[1] of AS:

$$t = -\frac{a-z}{\bar{a}-\bar{z}} = \cdots = -1$$

which indeed equals the slope of BC $(= d^2 = (-i)^2 = -1)$.

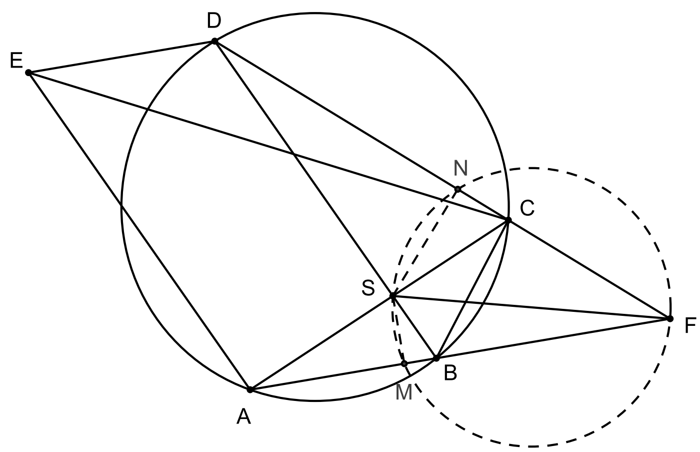
[1]Slope of the line with the equation $z + t\bar{z} = s$ is t.

Problem T-6. *Let A, B, C, D, E be points such that $ABCD$ is a cyclic quadrilateral and $ABDE$ is a parallelogram. The diagonals AC and BD intersect at S and the rays AB and DC intersect at F. Prove that $\angle AFS = \angle ECD$.*

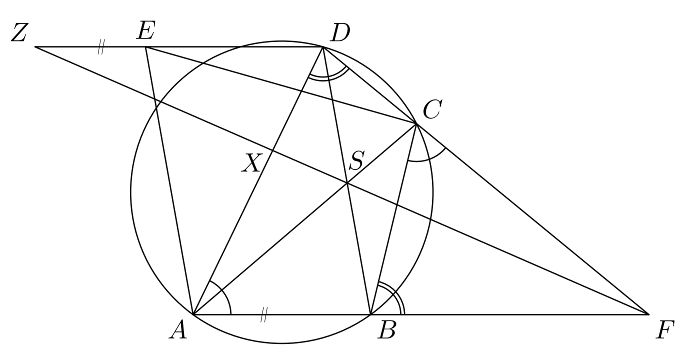
Solution. Let M and N be the feet of perpendicular from S to AB and CD, respectively. Then $SMFN$ is cyclic since it has two opposite right angles. Therefore $\angle AFS = \angle MFS = \angle MNS$. We need to prove $\angle MNS = \angle ECD$. This will follow from similarity of triangles MSN and EDC. Since $ABCD$ is cyclic, triangles ABS and DCS are similar. Lines SM and SN are the corresponding altitudes, so $SM : SN = AB : CD = ED : CD$. Also,

$$\angle MSN = 180^\circ - \angle AFD = \angle EDF = \angle EDC$$

and therefore, $\triangle MSN \sim \triangle EDC$ as claimed.



Solution 2. Let FS intersect the lines AD and DE in the points X and Z respectively and denote $\alpha = \angle BAD$, $\delta = \angle ADF$, $AB = a$, and $CD = c$. It is sufficient to prove that the triangles CDE and ZDF are similar, because then $\angle ECD = \angle DZF = \angle AFS$. These triangles have a common angle, we have to prove that $ZD : FD = c : a$.



The sine law in the triangle BFC gives $CF : BF = \sin\delta : \sin\alpha$, since

$$\angle FBC = 180^\circ - \angle ABC = \angle CDA = \delta \quad \text{and} \quad \angle FCB = 180^\circ - \angle BCD = \angle BAD = \alpha.$$

The Ceva's theorem for the triangle AFD and the point S then gives

$$1 = \frac{DX}{AX} \cdot \frac{AB}{BF} \cdot \frac{CF}{DC} = \frac{DX}{AX} \cdot \frac{a}{c} \cdot \frac{\sin\delta}{\sin\alpha}. \qquad (1)$$

From the similitude of triangles AFX and DZX we have $ZD = AF \cdot DX/AX$, and from the sine law in the triangle AFD we have $AF = FD \cdot \sin\delta / \sin\alpha$. Therefore the desired ratio $ZD : FD$ equals

$$\frac{ZD}{FD} = \frac{AF}{FD} \cdot \frac{DX}{AX} = \frac{\sin\delta}{\sin\alpha} \cdot \frac{DX}{AX} = \frac{c}{a},$$

where (1) is used in the last equality.

Solution 3. Let G be the intersection of lines AE and CD and let T be such point on AG that $\angle BFS = \angle TFD$. Then we have to prove that $FT \parallel CE$ or equivalently, that $|CG|/|FG| = |EG|/|TG|$.

Since $ABDE$ is a parallelogram, the triangles EDG and AFG are similar, therefore

$$|EG| = \frac{|AG| \cdot |ED|}{|AF|} = \frac{(|BD| + |EG|) \cdot |AB|}{|AF|},$$

so

$$|EG| = \frac{|BD| \cdot |AB|}{|BF|}.$$

Similarly, since the triangles BFD and EDG are similar, we obtain

$$|DG| = \frac{|FD| \cdot |AB|}{|BF|}.$$

Since $ABCD$ is cyclic, the triangles BFC and DFA are similar, so $|AF| \cdot |BF| = |CF| \cdot |DF|$. Now we compute

$$\begin{aligned} |CG| = |DG| + |CD| &= \frac{|FD| \cdot |AB|}{|BF|} + |FD| - |FC| = \\ &= \frac{|FD| \cdot |AF|}{|BF|} - \frac{|AF| \cdot |BF|}{|FD|} = |FA| \cdot \left(\frac{|DF|}{|BF|} - \frac{|BF|}{|DF|} \right) \end{aligned}$$

and

$$|FG| = |DG| + |FD| = \frac{|FD| \cdot |AF|}{|BF|},$$

so

$$\frac{|CG|}{|FG|} = 1 - \left(\frac{|BF|}{|DF|} \right)^2.$$

Now we will compute also the ratio $|EG|/|TG|$. By the construction of T we have $\angle AFT = \angle SFC$. Moreover, since $ABCD$ is cyclic and $AT \parallel BD$, we have also

$$\angle TAF = \angle TAD + \angle DAF = \angle ADB + \angle BCF = \angle ACB + \angle BCF = \angle ACF,$$

so the triangles AFT and CFS are similar. Since BFC and DFA are also similar, we get $|BC|/|AD| = |CF|/|AF| = |CS|/|AT|$. Since the angles $\angle DAT$ and SCB are also equal, the triangles ADT and CBS are similar. Then $\angle TDA = \angle SBC = \angle DAC$, so $DT \parallel AS$.

Then $ASDT$ is a parallelogram and the triangles TDE and SAB are congruent. Therefore $|TG| = |EG| + |TE| = |EG| + |BS|$. The triangles ABS and DCS are similar, therefore

$$\frac{|BS|}{|AB|} = \frac{|CS|}{|CD|} = \frac{|AC| - |AS|}{|CD|} \quad \text{and} \quad |AS| = |AC| - \frac{|BS| \cdot |CD|}{|AB|}.$$

Since the triangles BFS and DFT are similar, we have

$$\frac{|BS|}{|BF|} = \frac{|DT|}{|DF|} = \frac{|AS|}{|DF|} = \frac{|AC|}{|DF|} - \frac{|BS| \cdot |CD|}{|AB| \cdot |DF|},$$

so

$$|BS| = \frac{|AC| \cdot |AB| \cdot |BF|}{|AB| \cdot |DF| + |BF| \cdot |CD|}.$$

We use also the similarities of AFC and DFB and of AFD and CFB and obtain

$$|BS| = \frac{|BD| \cdot |CF| \cdot |AB| \cdot |DF|}{|AB| \cdot |DF|^2 + |BF| \cdot |DF|^2 - |BF|^2 \cdot |AF|} = \frac{|BD| \cdot |AB| \cdot |BF|}{|DF|^2 - |BF|^2}$$

and

$$|TG| = |EG| + |BS| = \frac{|BD| \cdot |AB| \cdot |DF|^2}{|BF| \cdot (|DF|^2 - |BF|^2)}.$$

Finally we can compute

$$\frac{|EG|}{|TG|} = \frac{|DF|^2 - |BF|^2}{|DF|^2} = 1 - \left(\frac{|BF|}{|DF|}\right)^2 = \frac{|CG|}{|FG|},$$

which we had to prove.

Problem T-7. *For a nonnegative integer n, define a_n to be the positive integer with decimal representation*

$$1\underbrace{0\ldots0}_{n}2\underbrace{0\ldots0}_{n}2\underbrace{0\ldots0}_{n}1.$$

Prove that $a_n/3$ is always the sum of two positive perfect cubes but never the sum of two perfect squares.

Solution. First we prove that $a_n/3$ is never the sum of two perfect squares. Note that perfect squares give only remainders 0 and 1 when divided by four; therefore, integers expressible as the sum of two squares give only remainders 0, 1, and 2. On the other hand, the number $a_n/3$ gives remainder 3 because a_n gives remainder 1; hence it cannot be expressed as the sum of two perfect squares.[2]

After some experimentation, one finds the formula

$$\frac{a_n}{3} = \left(\frac{10^{n+1}+2}{3}\right)^3 + \left(\frac{2\cdot 10^{n+1}+1}{3}\right)^3.$$

This follows from the fact that $a_n = 10^{3n+3} + 2\cdot 10^{2n+2} + 2\cdot 10^{n+1} + 1$. Both the numbers in brackets are integers since $10^{n+1} \equiv 1 \pmod 3$. Thus $a_n/3$ can be expressed as the sum of two perfect cubes.

[2]Another way to look at $a_n/3$ modulo 4 is to note that it always ends with 67.

Problem T-8. *We are given a positive integer n which is not a power of 2. Show that there exists a positive integer m with the following two properties:*

(i) m is the product of two consecutive positive integers;

(ii) the decimal representation of m consists of two identical blocks of n digits.

Solution. First we prove a lemma.

Lemma. *Let x and k be integers greater than 2. If k is odd then the number x^k+1 is the product of two coprime numbers.*

Proof. Let $m = gcd(x+1,k)$. There is a polynomial $Q(x) \in \mathbb{Z}[x]$ such that

$$x^k+1 = (x+1)(x^{k-1} - x^{k-2} + \cdots + x^2 - x + 1) = (x+1)\big((x+1)Q(x)+k\big).$$

Then

$$x^k+1 = \big((x+1)m\big) \cdot \left(\frac{(x+1)Q(x)}{m} + \frac{k}{m}\right)$$

gives the required product because

$$\frac{(x+1)Q(x)}{m} + \frac{k}{m} = \frac{x^k+1}{(x+1)m} \geq \frac{x^3+1}{(x+1)^2} > 1. \quad \square$$

The numbers 1 and 2 are powers of two, hence we may assume that $n \geq 3$. Since n is not a power of two, it has an odd divisor greater than one; therefore, according to our lemma,[3] there are coprime numbers a and b such that

$$10^n + 1 = ab.$$

Our task is to prove that there are numbers t and s such that

$$m = (10^n+1)t = abt = s(s-1).$$

First, we show that there is a positive integer s divisible by a which satisfies $s \equiv 1 \pmod{b}$. Consider the numbers $0, a, 2a, \ldots, (b-1)a$. These numbers give mutually different remainders modulo b since a and b are coprime. Therefore, one of them gives remainder 1 and we take s to be this number.

Similarly we can pick a number s' divisible by b which satisfies $s' \equiv 1 \pmod{a}$. The numbers s and s' are positive and smaller than 10^n. Therefore, $s(s-1)$ and $s'(s'-1)$ are both divisible by ab and smaller than 10^{2n}. Moreover, $s + s' \equiv 1 \pmod{ab}$. The number $s + s'$ is greater than 1 and smaller than $2 \cdot 10^n$. Hence $s + s' = ab + 1$. Therefore, one of the numbers s and s' is greater than $5 \cdot 10^{n-1}$. Then one of the numbers $s(s-1)$ or $s'(s'-1)$ is greater than $25 \cdot 10^{2n-2}$, thus it has $2n$ digits. This number has all the required properties.

Comment. Instead od using congruences we can also look at Diophantine equations $ax = by + 1$ and $ax = by - 1$. Both have solutions with $0 < x < b$ and $0 < y < a$ and $xy > 10^{n-1}$ for one of them.

[3] We can avoid using the lemma by exploiting the Mihailescu's theorem, first known as Catalan's Conjecture; it was proved in 2002. It says that the only solution of the equation $x^a - y^b = 1$ in positive integers greater than one is $3^2 - 2^3$. This implies that if $10^n + 1$ is a power of a prime then it is a prime. This cannot happen since n has an odd divisor.