language: English

4^{th} Middle European Mathematical Olympiad

TEAM COMPETITION
12^{th} SEPTEMBER, 2010

Problem T-1.
Three strictly increasing sequences

$$a_1, a_2, a_3, \ldots, \qquad b_1, b_2, b_3, \ldots, \qquad c_1, c_2, c_3, \ldots$$

of positive integers are given. Every positive integer belongs to exactly one of the three sequences. For every positive integer n, the following conditions hold:

(i) $c_{a_n} = b_n + 1$;

(ii) $a_{n+1} > b_n$;

(iii) the number $c_{n+1}c_n - (n+1)c_{n+1} - nc_n$ is even.

Find a_{2010}, b_{2010}, and c_{2010}.

Problem T-2.
For each integer $n \geq 2$, determine the largest real constant C_n such that for all positive real numbers $a_1, \ldots, a_n$, we have

$$\frac{a_1^2 + \cdots + a_n^2}{n} \geq \left(\frac{a_1 + \cdots + a_n}{n}\right)^2 + C_n \cdot (a_1 - a_n)^2.$$

Problem T-3.
In each vertex of a regular n-gon there is a fortress. At the same moment each fortress shoots at one of the two nearest fortresses and hits it. The *result of the shooting* is the set of the hit fortresses; we do not distinguish whether a fortress was hit once or twice. Let $P(n)$ be the number of possible results of the shooting. Prove that for every positive integer $k \geq 3$, $P(k)$ and $P(k+1)$ are relatively prime.

Problem T-4.
Let n be a positive integer. A square $ABCD$ is partitioned into n^2 unit squares. Each of them is divided into two triangles by the diagonal parallel to BD. Some of the vertices of the unit squares are colored red in such a way that each of these $2n^2$ triangles contains at least one red vertex. Find the least number of red vertices.

Problem T-5.
The incircle of the triangle ABC touches the sides BC, CA, and AB in the points D, E, and F, respectively. Let K be the point symmetric to D with respect to the incenter. The lines DE and FK intersect at S. Prove that AS is parallel to BC.

Problem T-6.
Let A, B, C, D, E be points such that $ABCD$ is a cyclic quadrilateral and $ABDE$ is a parallelogram. The diagonals AC and BD intersect at S and the rays AB and DC intersect at F. Prove that $\angle AFS = \angle ECD$.

Problem T-7.
For a nonnegative integer n, define a_n to be the positive integer with decimal representation

$$1\underbrace{0\ldots0}_{n}2\underbrace{0\ldots0}_{n}2\underbrace{0\ldots0}_{n}1.$$

Prove that $a_n/3$ is always the sum of two positive perfect cubes but never the sum of two perfect squares.

Problem T-8.
We are given a positive integer n which is not a power of 2. Show that there exists a positive integer m with the following two properties:

(i) m is the product of two consecutive positive integers;

(ii) the decimal representation of m consists of two identical blocks of n digits.

Time: 5 hours
Time for questions: 45 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.